Diagonalization and eigenvalues

There are lots of real-life systems that evolve over time that can be modelled with linear algebra. We will see that diagonalization helps us describe these systems.

Example: bird population

For this example, we assume # of males = # of females and we only count females.

Three assumptions governing their population growth:

1. # of juvenile females hatched in a year = 2 · # adult females alive the previous year

   (reproduction rate = 2)

2. $\frac{1}{2}$ adult females survive to the next year

   (adult survival rate = $\frac{1}{2}$)

3. $\frac{1}{4}$ juveniles survive to adulthood. (juvenile survival rate = $\frac{1}{4}$)

If there are 100 adult females, 40 juvenile females, what is the population in k years?

a_k = # adults in k years

j_k = # juveniles in k years

Total pop = $a_k + j_k$

The assumptions give us recursive equations:

$$a_{k+1} = \frac{1}{2} a_k + \frac{1}{4} j_k$$
$$j_{k+1} = 2a_k$$

Set $\vec{v}_k = \begin{bmatrix} a_k \\ j_k \end{bmatrix}$.

Then $\vec{v}_{k+1} = \begin{bmatrix} \frac{1}{2} a_k + \frac{1}{4} j_k \\ 2a_k \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 2 & 0 \end{bmatrix}}_{A} \vec{v}_k$

so
$$\vec{v}_1 = A\vec{v}_0$$
$$\vec{v}_2 = A\vec{v}_1 = A^2\vec{v}_0$$
$$\vdots$$
$$\vec{v}_k = A^k\vec{v}_0.$$

We know $\vec{v}_0 = \begin{bmatrix} 100 \\ 40 \end{bmatrix}$, so to find $\vec{v}_k$, we need to compute A^k for $k > 0$. We'll come back to this once we have more technical machinery.

Def: If A is a square matrix, then a sequence $\vec{v}_0, \vec{v}_1, \vec{v}_2, \ldots$ of vectors is called a <u>linear dynamical system</u> if $\vec{v}_0$ is known and

$$\vec{v}_{k+1} = A\vec{v}_k \text{ for each } k \geq 0.$$

These conditions are called a <u>matrix recurrence</u> and just like in the above example, they imply $\vec{v}_k = A^k \vec{v}_0$

How do we compute A^k?

Idea: D is a diagonal matrix if it is 0 away from the diagonal. D^k is easy to compute.

To compute A^k, we diagonalize it, i.e. we find an invertible matrix P such that $P^{-1}AP = D$, a diagonal matrix.

Then $A = PDP^{-1}$, so $A^k = \underbrace{(PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\cdots(\cancel{P}DP^{-1})}_{k \text{ times}}$

$= PD^kP^{-1}$, which is easy to compute.

We need to learn how to find the matrix P. We do this by first computing the eigenvalues of A:

Eigenvalues + Eigenvectors

Def: If A is an $n \times n$ matrix, a number λ is an eigenvalue of A if $A\vec{x} = \lambda\vec{x}$ for some vector $\vec{x} \neq \vec{0}$ in $\mathbb{R}^n$. $\vec{x}$ is called an eigenvector corresponding to λ, or a λ-eigenvector.

Ex: Let $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.

Notice that $\underbrace{\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \underbrace{2}_{\lambda} \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{\vec{x}}$

So $\lambda = 2$ is an eigenvalue of A with corresponding eigenvector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

How do we find eigenvalues/eigenvectors?

Note that $\lambda \vec{x} = \begin{bmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \ddots \ \lambda \end{bmatrix} \vec{x} = \lambda I \vec{x}$.

So $A\vec{x} = \lambda \vec{x}$ is the same as $A\vec{x} = \lambda I \vec{x}$.

So λ is an eigenvalue if

$$\lambda I \vec{x} - A\vec{x} = \vec{0} \quad \text{for some } \vec{x} \neq 0.$$

That is, if $(\lambda I - A)\vec{x} = \vec{0}$ has a nontrivial solution.

This happens if and only if $\lambda I - A$ is not invertible, i.e.

$$\det(\lambda I - A) \neq 0.$$

Def: If A is an $n \times n$ matrix, the <u>characteristic polynomial</u> of A is

$$c_A(x) = \det(xI - A).$$

(degree n)

Then we can see that λ is an eigenvalue if and only if $c_A(\lambda) = 0$, i.e. if λ is a <u>root</u> of $c_A(x)$. This gives us the following:

Thm: Let A be an $n \times n$ matrix.

1. The eigenvalues of A are the roots of the characteristic polynomial $c_A(x)$ of A.

2. If λ is an eigenvalue of A, then the λ-eigenvectors are the nonzero solutions to the system

$$(\lambda I - A)\vec{x} = \vec{0}.$$

Ex: Let's go back to the matrix

$$A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}.$$

$$c_A(x) = \det(xI - A) = \det\left(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}\right)$$

$$= \det \begin{bmatrix} x-4 & 3 \\ -2 & x+1 \end{bmatrix}$$

$$= (x-4)(x+1) - (3)(-2)$$
$$= x^2 - 3x - 4 + 6$$
$$= x^2 - 3x + 2$$
$$= (x-1)(x-2)$$

This has two roots: 1, 2. So A has eigenvalues

$$\lambda_1 = 1$$

$$\lambda_2 = 2$$

First, we find the λ_1-eigenvectors:

We solve the system

$$(\lambda_1 \cdot I - A)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}\right)\vec{x} = \vec{0}$$

$$= \begin{bmatrix} -3 & 3 \\ -2 & 2 \end{bmatrix}\vec{x} = \vec{0}$$

This has solution $-1x_1 + 1x_2 = 0$.

Setting $x_2 = t$, the general solution is

$$\begin{aligned} x_1 &= 1 \\ x_2 &= t \end{aligned}, \quad \text{or} \quad \vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So the eigenvectors corresponding to the eigenvalue 1 are

$$t\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad t \neq 0$$

For the 2-eigenvectors, we solve

$$\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}\right)\vec{x} = \vec{0}$$

$$\begin{bmatrix} -2 & 3 \\ -2 & 3 \end{bmatrix}\vec{x} = \vec{0}$$

$$\Rightarrow -2x_1 + 3x_2 = 0 \Rightarrow x_1 = \frac{3}{2}x_2$$

So 2-eigenvectors are $t\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$, $t \neq 0$.

Note: For each eigenvalue λ of a matrix A, there will be infinitely many λ-eigenvectors, all nontrivial solutions of $(\lambda I - A)\vec{x} = \vec{0}$.

The eigenvectors will be linear combinations of basic solutions, called <u>basic eigenvectors</u> corresponding to λ.

Ex:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

characteristic polynomial $= c_A(x) = \det\begin{bmatrix} x-2 & 0 & 0 \\ -1 & x-2 & 1 \\ -1 & -3 & x+2 \end{bmatrix}$

$$= (x-2)\Big((x-2)(x+2) - (1)(-3)\Big)$$
$$= (x-2)(x^2 - 4 + 3)$$
$$= (x-2)(x-1)(x+1)$$

So A has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = -1$

For $\lambda_1 = 2$:

$$\lambda_1 I - A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & -3 & 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -3 & 4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ -1 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{array}$$

solution: $\begin{array}{l} x_1 = t \\ x_2 = t \\ x_3 = t \end{array} \Rightarrow \lambda_1$-eigenvectors : $t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $t \neq 0$.

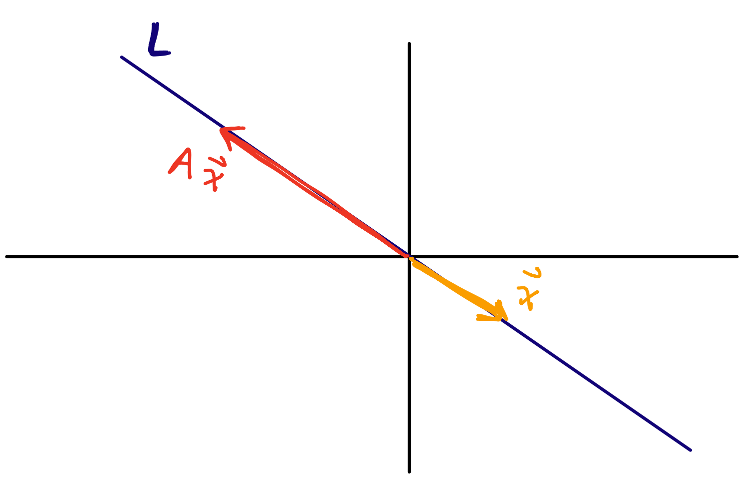
Practice problem: Find eigenvectors for remaining 2 eigenvalues $\lambda_2 = 1$, $\lambda_3 = -1$.

Note: $xI - A^T = (xI - A)^T$, so A^T and A have same eigenvalues.

A-invariance

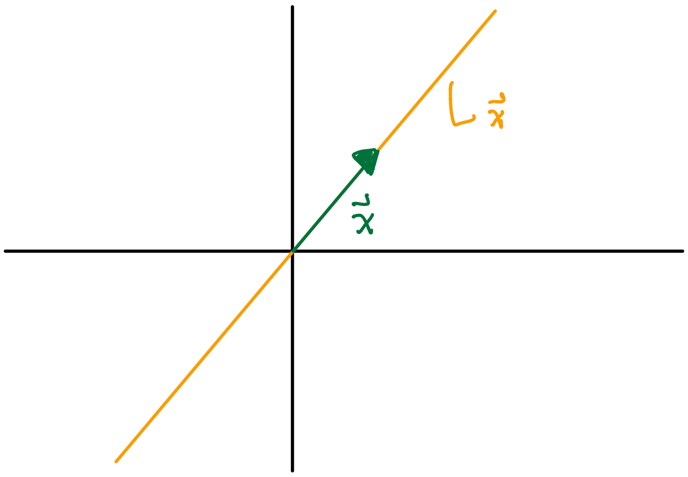
Let A be a 2×2 matrix. A line L through the origin in $\mathbb{R}^2$ is called <u>A-invariant</u> if $A\vec{x}$ is in L whenever $\vec{x}$ is in L.

Geometrically:



How does this relate to eigenvectors?

Let $\vec{x} \neq \vec{0}$ be any nonzero vector in $\mathbb{R}^2$. Let $L_{\vec{x}}$ be the unique line through the origin containing $\vec{x}$.



$L_{\vec{x}}$ consists of all scalar multiples of $\vec{x}$. That is,

$$L_{\vec{x}} = \{ t\vec{x} \mid t \text{ is in } \mathbb{R} \}$$

Suppose $\vec{x}$ is an eigenvector of A, so $A\vec{x} = \lambda \vec{x}$, some scalar λ. Then for any $t\vec{x}$ in the line $L_{\vec{x}}$, we have

$$A(t\vec{x}) = t(A\vec{x}) = \underbrace{(t\lambda)}_{\text{scalar}}\vec{x},$$ which is again in $L_{\vec{x}}$.

So $L_{\vec{x}}$ is A-invariant.

The converse holds as well: if $L_{\vec{x}}$ is A-invariant, then $A\vec{x}$ is in $L_{\vec{x}}$, so $A\vec{x} = t\vec{x}$, some t, so $\vec{x}$ is an eigenvector for A, w/ eigenvalue t.

Thus, we've proved the following theorem:

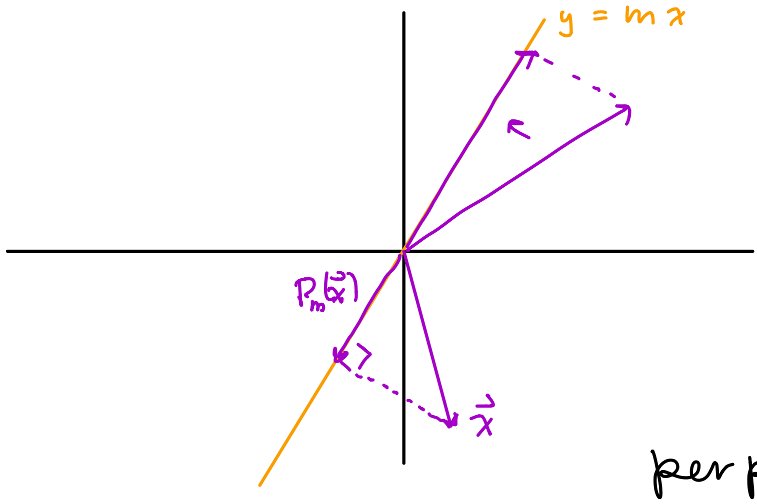
Theorem: Let A be a 2×2 matrix, $\vec{x} \neq \vec{0}$ a vector in $\mathbb{R}^2$, and $L_{\vec{x}}$ the line through the origin containing $\vec{x}$. Then

$\vec{x}$ is an eigenvector of A if and only if $L_{\vec{x}}$ is A-invariant.

Ex: ① Let $R_{\pi/2}: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which is CCW rotation by $\pi/2$ with corresponding matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

No line is A-invariant, since it sends every vector to a different line. Thus, A has no eigenvalues

② Let $P_m: \mathbb{R}^2 \to \mathbb{R}^2$ correspond to projection onto the line $y = mx$, A the corresponding matrix.



The only vectors that stay in the same line are those in the line $y = mx$, OR in the line perpendicular to $y = mx$, i.e. $y = \frac{-1}{m} x$ (assuming $m \neq 0$)

In the first case, this means that if $\vec{x} = \begin{bmatrix} 1 \\ m \end{bmatrix}$, Then $L_{\vec{x}}$ is A-invariant.

Moreover, $A \begin{bmatrix} 1 \\ m \end{bmatrix} = \begin{bmatrix} 1 \\ m \end{bmatrix}$ (projection onto itself)

$t \begin{bmatrix} 1 \\ m \end{bmatrix}$ are eigenvectors w/ corresponding eigenvalue 1.

In the second case, all the vectors in $y = \frac{-1}{m}x$ get sent to $\vec{0}$, so the corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ -1/m \end{bmatrix} t \,, \quad \text{or, equivalently} \quad \begin{bmatrix} m \\ -1 \end{bmatrix} t \,, \quad t \neq 0.$$

In this case $A \begin{bmatrix} m \\ -1 \end{bmatrix} = \vec{0} = 0 \begin{bmatrix} m \\ -1 \end{bmatrix}$, so the corresponding eigenvalue is 0.